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Research Article

THE EFFECT OF TRANSLATION ON THE APPROXIMATED FIRST ORDER POLARIZATION TENSOR OF SPHERE AND CUBE

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ABSTRACT

Throughout this paper, the translation effect on the first order polarization tensor approximation for different type of objects will be highlighted. Numerical integration involving quadratic element as well as linear element for polarization tensor approximation will be presented. Here, we used different positions of an object of fixed size and conductivity when computing the first order polarization tensor. From the numerical results of computed first order polarization tensor, the convergence for every translation is observed. Moreover, discretization of the geometric objects into triangular meshes was done by using meshing software called NETGEN mesh generator while for the numerical computation, MATLAB software was used. We found that the translation has no effect on the approximated first order PT for sphere and cube after we have computed the first order PT for both geometries with a few center of masses. The numerical results of approximated first order polarization tensor is plotted by comparing the numerical results with analytical solution provided.

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INTRODUCTION

In the applications of electric or electromagnetic such as medical imaging and metal detection, some properties of an electrical conducting object will be described for classification of object purposes. These properties include the material (or the conductivity) of the object, size, shape and also the orientation of the object. The object could be a tumor with larger value of conductivity than normal tissues in medical imaging (1) or a fish hook avoided by electrosensing fish (2-6) whereas, in metal detection, the object could be a buried landmine or a gun carried by a person when walking under metal detector (7-13).

There are probably many approaches exist that can be used to describe these objects and their properties. One way to characterize these objects is to implement the Polarization Tensor (PT) terminology, as can be seen in the previous listed references. This approach seems to offer lower computational cost as it does not require for example, complete image of the object to describe the object.

In applied mathematics, Ammari and Kang (14), they have shown that the leading term of asymptotic expansion that represent the disturbance in electrical field because of the existence of object with conductivity is called Generalized Polarization Tensor. Given the conductivity and geometry of the object, by solving system of integral equation, the associated GPT in the matrix form or in some literatures known as PT for object with specific conductivity can be determined. In other words, given GPT due to the presence of a known or unknown object, the geometry and conductivity of the related object can be described or predicted (15-17).

PT for an object with some conductivity when presented in the electrical field does not give any information about the specific location of the object. Other methods must be integrated with PT to locate the object when reconstructing the image of the object (14), for example. Similarly, a person can carry a gun at any parts of the body before entering a metal detector, and when the gun is detected, the body must be searched manually by a security officer.

Therefore, the translating effect of the object on the computation of its first order PT will be investigated throughout this study. Theoretically, as will be stated later on, the PT of an object with specific conductivity are independent of the position of the object. Therefore, our numerical results on computing the first order PT of an object at different locations must be the same or at least, have only very small differences between them. This will justify whether our numerical method produces a good approximation to the PT. Besides, for some cases, computing the PT of an object at different locations will also help to verify that the resulting PT is correct.

The paper will is organized as follows. First, the background of mathematical term of the PT related to the disturbance in electric field because of the existence of object with specific conductivity will be reviewed in the next section. From the mathematical background, we will explain the translation of the PT and methods used to generate the numerical results of each translation. Then, numerical results showing the effect of translation on the approximation of the PT will be presented in graphical form. Lastly, discussion and some conclusions about the numerical results obtained are presented.

FIRST ORDER POLARIZATION TENSOR FORMULATION

Polarization tensor as considered here actually initiates from a transmission problem of the inverse conductivity equation which has being reviewed by a numbers of literatures. Ammari and Kang (14) consider a Lipschitz bounded domain B in $\mathbb{R}^3$ where the origin O is inside the boundary domain B and k will represent the conductivity of domain B. The condition of the conductivity k must satisfy the condition where, $0 < k \neq 1 < +\infty$. By assuming harmonic function, which denotes as H is in $\mathbb{R}^3$ and the solution of the problem in (1) is u, then

$$\begin{cases} \operatorname{div}(1+(k-1)\chi(B)\operatorname{grad}(u))=0 \text{ in } \mathbb{R}^3 \\ u(x)-H(x)=O(|x|^{-2}) \text{ as } |x|\to\infty \end{cases} \tag{1}$$

where χ represents the characteristic function of B. There are numerous industrial application that employ the mathematical formulation in (1); for example in medical imaging where we can see from Electrical Impedance Tomography (EIT) system, material sciences as well as in detection of landmine (1, 14, 18).

Far-field expansion that represent PT has been introduced by Ammari and Kang (14) where it yields to

$$(u-H)(x)=\sum_{|i|,|j|=1}^{+\infty}\frac{(-1)^{|i|}}{i!\,j!}\partial_x^i\Gamma(x)M_{ij}(k,B)\partial^j H(0) \text{ as } |x|\to+\infty \tag{2}$$

for i, j that denotes the multi indices, while Γ is the fundamental solution of the Laplacian. $M_{ij}(k,B)$ is the generalized polarization tensor (GPT) or can be simply called as PT. Since GPT can show the conductivity distribution of an object, therefore, physicist usually assigned GPT as the dipole in electromagnetic applications. In this case, (2) represent the perturbation on the electrical field u because of the presence of a conducting object B.

Furthermore, according to Ammari and Kang, the GPT in (2) can also be defined by system of integral equations over the boundary of B which is

$$M_{ij}(k,B)=\int_{\partial B} y^j\phi_i(y)d\sigma(y) \tag{3}$$

where $\phi_i(y)$ is formulated as

$$\phi_i(y)=\left(\lambda I-K_B^*\right)^{-1}\left(\nu_x\cdot\nabla x^i\right)(y) \tag{4}$$

for $x, y\in\partial B$, I is the identity while ν_x is the outward normal vector of x to the boundary ∂B. In this case, the definition of λ is $\lambda=(k+1)/2(k-1)$ which contain the conductivity, k. Singular integral operator, K_B^* associated with Cauchy principal value $P.V.$ is given and yield to

$$\kappa_B^*\phi(x)=\frac{1}{4\pi}\text{p.v.}\int_{\partial B}\frac{(x-y)\cdot\nu_x}{|x-y|^3}\phi(y)d\sigma(y)\,. \tag{5}$$

In order to directly compute the PT of B for conductivity k, where k must satisfy $0<k\neq 1<+\infty$, we can simply determine it by substituting the conductivity and B to equation (3), (4) and (5).

In this work, we restrict our investigation on the first order PT, when the multi indices of GPT is $|i|=|j|=1$. Thus, by construction, the first order PT is represented by matrix of 3 by 3. If B is a sphere of volume $|B|$, then from the analytical formula that has been

derived and introduced by Ammari and Kang (14), its first order polarization tensor will yield to

$$M(k,B) = (k-1)\,|B| \begin{bmatrix} 3/(2+k) & 0 & 0 \\ 0 & 3/(2+k) & 0 \\ 0 & 0 & 3/(2+k) \end{bmatrix}. \quad (6)$$

In the next section, we will revise previous theoretical results which suggest that the first order PT is independent of the position of *B*. After that, we explain our numerical method for the first order PT approximation for sphere and cube geometries of constant density at specified conductivity but with different center of mass.

METHODOLOGY

In the previous section, it is required that the origin must be in *B* but not the center of mass for *B*. So, the center of mass for *B* can be anywhere when its first order PT is computed. The following proposition is considered from the study of Ammari and Kang (14), which explains the first order PT is independent of the position of object *B*.

Proposition 1

Let $M(k,B)$ be the first order polarization tensor for an object *B* at conductivity *k*, where $0 < k \neq 1 < +\infty$ and $\mathbf{z}$ be a translation vector. If $M(k,\mathbf{z}+B)$ is the first order PT for *B* after *B* is translated by $\mathbf{z}$ then

$$M(k,\mathbf{z}+B) = M(k,B).$$

The above proposition specifically tells us that if the center of mass of *B* is changed, then its first order PT will remain the same. If we regard the first order PT as a physical parameter of *B* such as surface area, volume or density, the first order PT is independent of position of *B*. This information is useful to us as it can help us to verify whether the first order PT of *B* that is numerically computed is correct or not; thus, we can compare the first order PT for *B*, computed at two different center of mass and ensure that they are the same.

In general, the first order PT can only be computed by numerical method, except when *B* is a sphere. In this case, finite element approach will be employed to approximate the first order PT for sphere and cube based on equations (3), (4) and (5). Here, to numerically compute first order PT for sphere and cube geometries, each object will be initially created in the software Netgen (19) at a specified center of mass. After that, mesh of the object consisting of a set of triangular elements will be automatically created by the software. Since the integrals involved as given in (3) and (5) are surface integrals, the triangular elements generated are actually the surface elements of the object. The required nodes for each triangle given by Netgen will be exported to Matlab for computation of the PT by finite element method. For comparison in this study, quadratic and linear element which involve six and three nodes in each triangle will be used (20).

After generating the triangular elements for both objects, their first order PT will be computed at a specified conductivity k. The procedures when using linear elements are based on Khairuddin and Lionheart (21) and we refer to Reddy (20) when developing numerical algorithm with quadratic elements to approximate the first order PT. Then, we will increase the size of the mesh with Netgen and repeat the processes until satisfactory results are obtained. Usually, the size of mesh must be increased until the numerical computations of the first order PT converge to one value. The convergence first order PT will then be used as the convergence approximation of the first order PT. However, at the moment, approximating the first order PT is time consuming because of the large size for the mesh used. So, our results here might only be preliminary results.

For sphere, the error in approximating the first order PT can be computed by comparing the approximated first order PT with (6). Since the analytical solution of the first order PT for cube geometries has yet to be found, we cannot compute its error. However, we can still further investigate what happens to the first order PT approximation for both cube and sphere when the center of mass for both objects are changed. In order to achieve this, the center of mass for both sphere and cube are redefined in Netgen before the new meshes are created for computing the new first order PT but still at the same conductivity. Mesh for sphere and cube at a few center of masses will be considered in this study when approximating the first order PT for both objects.

NUMERICAL RESULTS AND DISCUSSION

For the purpose of first order PT approximation for sphere of radius 0.01, at a few center of masses, four coordinates are chosen and defined as the center in Netgen. The four chosen centers of mass for the sphere are $(0,0,0)$, $(0.01,0,0)$, $(0,0.01,0)$ and $(0,0,0.01)$. For each sphere with different center of mass, the meshes are then generated in Netgen and surprisingly, the size for the mesh for each sphere can still be the same. As the positions of each sphere are different from each other, the nodes for the triangular elements will be different although the total elements are the same. For the computation purposes, the sizes of the mesh for each sphere used are 620, 2480 and 9920.

Using the generated meshes, the first order PT for the sphere are numerically approximated at conductivity $k = 1.5$. After that, the error for each computation will be computed. If $\hat{M}$ is the approximated first order PT for sphere and M is the first order PT for sphere given by (6), then the error, e is given by

$$e = \frac{\| \hat{M} - M \|_2}{\| M \|_2}. \tag{7}$$

where for a 3×3 matrix A,

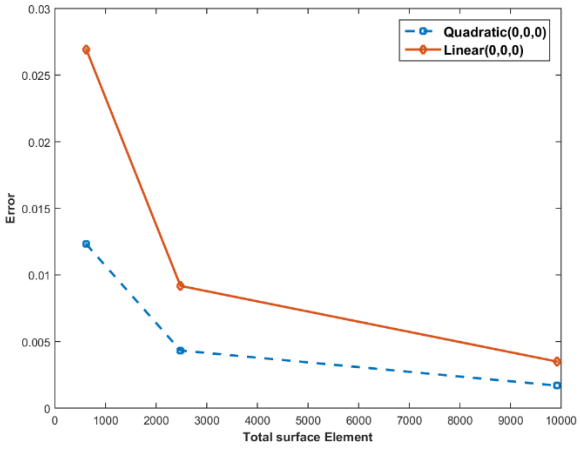
$$\| A \|_2 = \sqrt{\sum_{i=1}^{3}\sum_{j=1}^{3}\left(a_{ij}\right)^2} \tag{8}$$

and a_{ij} are the elements of the matrix A. Fig. 1 then shows the error against total surface elements of the approximated first order PT for sphere of radius 0.01 at conductivity, $k = 1.5$ when the center of mass for the sphere are respectively $(0,0,0)$, $(0.01,0,0)$, $(0,0.01,0)$ and $(0,0,0.01)$. As we can see in all graphs of Fig. 1, the error decreases as the total surface elements increase in approximating the first order PT for each sphere and the error when using quadratic elements for the sphere is smaller than the linear elements for each mesh, as expected.

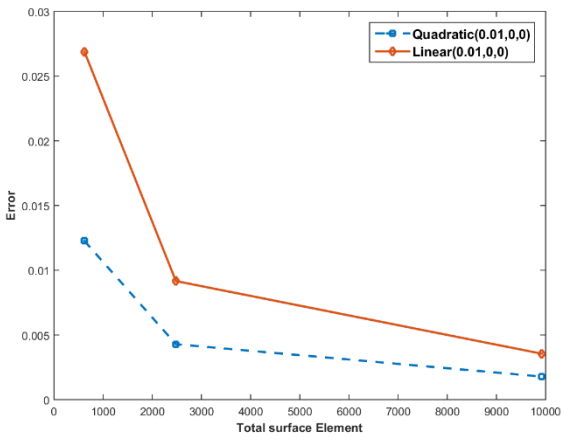
Now, in order to study the translation effect on the approximated first order PT, graphs in Fig. 2 are plotted. These graphs show the average of the elements, a for the approximated first order PT, $\hat{M}$ against the total surface elements for each mesh used. In this case, a is computed by the formula

$$a = \frac{1}{9}\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\hat{m}_{ij}\right)\right) \tag{9}$$

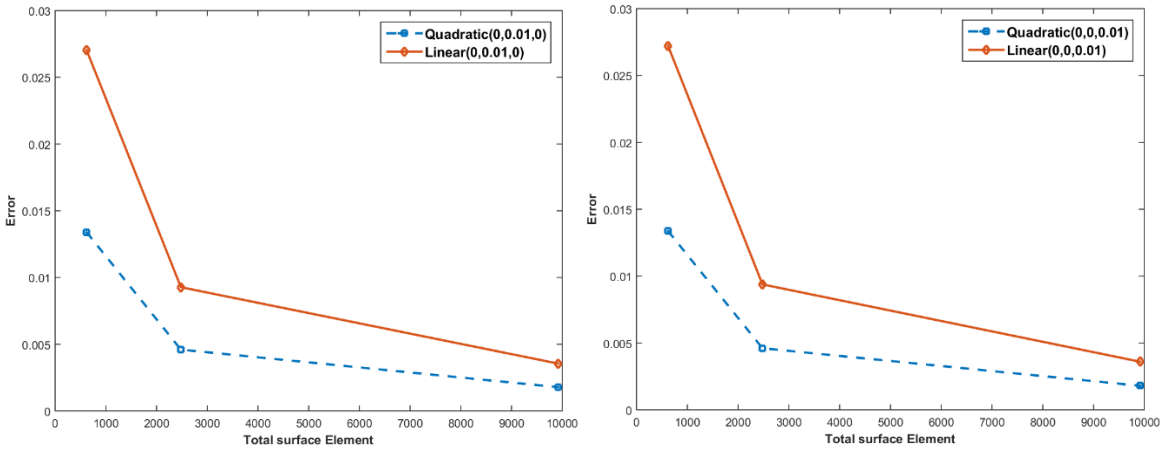
where $\hat{m}_{ij}$ are the elements of the matrix $\hat{M}$. Based on the graphs, we have noticed that a increases when M is computed either by linear or quadratic elements increased. Both average approach to the straight line which represents the average elements of the first order PT for the sphere computed based on the analytical formula (6). It is expected that, for each mesh, a computed based on M that is approximated by quadratic elements (Quad) are closer to the straight line than a computed based on M that is approximated by linear elements (Lin). Moreover, it is also observed that, for each mesh, the average elements of M are all the same when M are approximated by both linear or quadratic elements although the center of mass for the sphere are $(0,0,0)$, $(0.01,0,0)$, $(0,0.01,0)$ and $(0,0,0.01)$.



(a)



(b)



(c) (d)

Fig. 1. The error, e against the total surface elements in approximating the first order PT for sphere of radius 0.01 at conductivity $k = 1.5$ where the center of the mass for the sphere are respectively (a) $(0,0,0)$, (b) $(0.01,0,0)$, (c) $(0,0.01,0)$ and (d) $(0,0,0.01)$

Finally, Fig. 3 shows the average of the elements for the approximated first order PT for cube against the total surface elements for each mesh used. Here, the total surface elements or the size of the mesh for the cube generated automatically in Netgen are 192, 768, and 3072. In addition, the first order PT for the cube of size $0.04 \times 0.04 \times 0.04$, in this case, are approximated based on (3), (4) and (5) also at conductivity $k = 1.5$. However, only quadratic elements are used during the numerical approximation. Here, it is found that, for each mesh, the average elements of first order PT approximation for cube resulted in similar number of mesh although the center of mass for the cube are $(0,0,0)$, $(-0.02,0,0)$ and $(0.02,0.02,0.02)$.

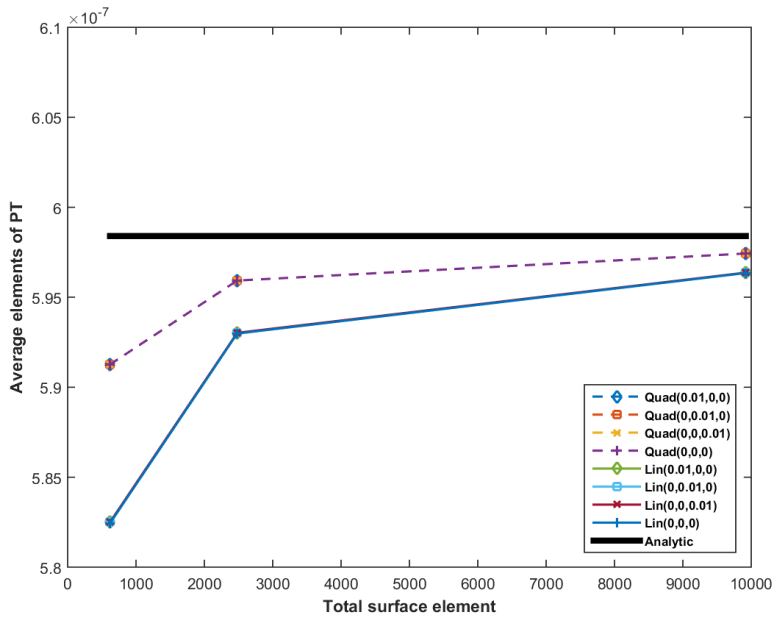


Fig. 2. The average elements, a of the approximated first order PT by both linear and quadratic elements against the total surface elements for the mesh. The first order PT for sphere of radius 0.01 at conductivity $k = 1.5$ are

approximated when the center of the mass for the sphere are respectively $(0,0,0)$, $(0.01,0,0)$, $(0,0.01,0)$ and $(0,0,0.01)$. The straight line represents the average elements of the first order PT for sphere computed based on the analytical formula (6).

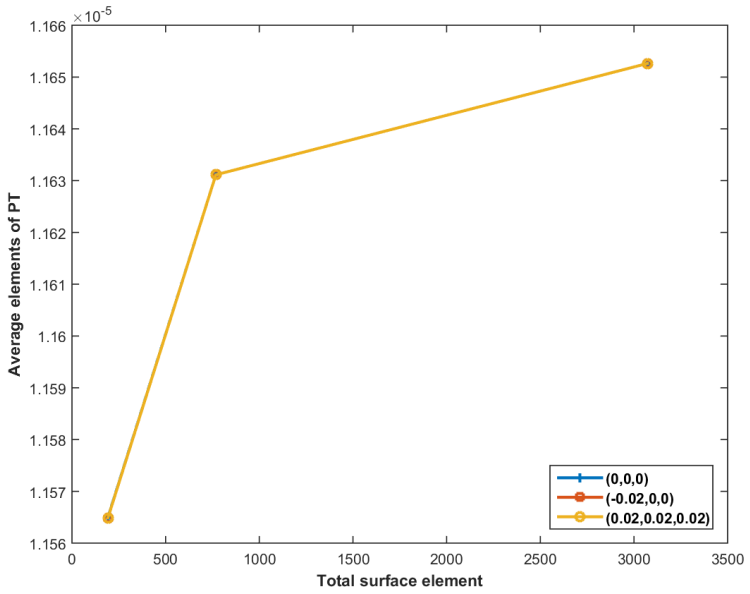


Fig. 3. The average elements, *a* of the approximated first order PT by quadratic elements for cube (size $0.04\times0.04\times0.04$ and conductivity $k=1.5$) against the total surface elements for the mesh, where, the center of the mass for the cube are respectively $(0,0,0)$, $(-0.02,0,0)$ and $(0.02,0.02,0.02)$.

CONCLUSION

In this study, we can observe that as the first order PT for sphere is evaluated either by linear or quadratic elements, the error of the computation is smaller if quadratic elements is used. The results are also true when the center of mass of the sphere are changed, as presented in Fig. 1. Moreover, the average elements of PT when it is approximated for both sphere or cube geometries are similar when the center of mass for both objects are relocated at a different place. This suggests that the approximated PT are also similar for both objects independent of their center of mass. Therefore, these results agree with the previous proposed theory stated in Proposition 1.

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