Fully Prime Double-Framed Soft Ordered Semigroups

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Abstract

It is well known that semigroups are basic structures in many applied branches like automata and formal languages, coding theory, finite state machines and others. Due to these possibilities of applications, semigroups and related structures are presently extensively investigated in soft theoretic (and in fuzzy theoretic) settings. The aim of this paper, is to introduce the concepts of double-framed soft ideals (briefly, DFS ideals) over an initial universe set U . The basic characterizations of this notion are studied. Different classes of ordered semigroups are characterized by using the notions of DFS l - and r -ideals over U . The notions of prime, and irreducible double-framed soft ideals (briefly, prime and irreducible DFS ideals) over U , are investigated and the related properties are highlighted. The class of those ordered semigroups for which each DFS ideal over U is prime are studied and the relationship between prime and irreducible DFS ideals are discussed.

Keywords: DFS Sets, DFS Ideals, Prime and Irreducible DFS Ideals, Semi-Simple Ordered Semigroups.

ARTICLE INFORMATION

Received: 10-Nov-2019

Revised: 26-Nov-2019

Accepted: 28-Dec-2019

DOI:

<https://doi.org/10.31580/ojst.v2i4.1063>

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INTRODUCTION

Molodtsov introduced the concept of a soft set [21], which can be used as a generic mathematical tool for dealing with uncertainties. The uncertainty appeared in economics, engineering, environmental science, medical science and social science and so many other applied sciences is too complicated to be solved by traditional mathematical framework. Molodtsov (1999) soft set theory [21] is a kind of new mathematical model for dealing with uncertainty from a parameterization point of view. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily applied to many different fields (e.g. see [1,3,4,7,9,10,12-15]). At present, the research works on the soft set theory in algebraic fields are progressing rapidly (see [22-25]).

Niovi Kehayopulu and others mainly studied the theory of ordered semi-groups (and poe-semigroups, ordered semigroups with greatest element e) (see [17,18]). She investigated many valuable results in this field and almost all the work on the ideal theory and fuzzy ideal theory of ordered semigroup have been developed [8,16]. Prime ideals, prime bi-ideals (respectively, fuzzy prime ideals and fuzzy prime bi-ideals) of semigroups and in other related structures have been studied by Shabir and others (see [26-28]).

Recently, Jun et al. further extended the notion of soft set into double-framed soft sets and defined double-framed soft subalgebra of a BCK/BCI-algebra and studied the related properties in [10]. Jun et al. also defined the concept of a double-framed soft ideal (briefly, DFS ideal) of a BCK/BCI-algebra and gave many valuable results for this theory.

In the present paper, we apply the idea given by Jun et al. in [10], to ordered semigroups and defined prime and irreducible DFS ideals of an ordered semigroup over U . The class of those ordered semigroups

are studied for which each DFS ideal is prime. In the last of this paper, the characterizations of these concepts are investigated in detail.

PRELIMINARIES

By an ordered semigroup, we mean a system $(S, \cdot, \leq)$ in which the following are satisfied:

(OS1) $(S, \cdot)$ is a semigroup, (OS2) $(S, \leq)$ is a poset,

(OS3) $x \leq y \Rightarrow ax \leq ay$ and $xa \leq ya$ for all $x, y, a \in S$.

Let $\emptyset \neq A \subseteq S$, we denote $[A]$ by $[A] := \{x \in S / x \leq a \text{ for some } a \in A\}$. If $A = \{a\}$, then we write $[a]$ instead of $\{[a]\}$. For any nonempty subsets A, B of S , we denote by $AB := \{AB / a \in A, b \in B\}$.

Definition 1 A non-empty subset A of an ordered semigroup S is called a sub-semigroup of S if $A^2 \subseteq A$.

In (Kehayopulu and Tesingelis [16]), defined that a nonempty subset A of an ordered semigroup S is called a left (resp., right) ideal of S if.

(1) $SA \subseteq A$ (resp., $AS \subseteq A$),

(2) If $b \in A, S \ni a \leq b$, then $a \in A$.

If A is both a left and a right ideal of S , then A is called a two-sided ideal or simply an ideal of S .

Definition 2 (cf. Khan [5]) Let S be an ordered semigroup and $a \in S$, then a is called idempotent if $a \leq a^2$. An ideal A of S is called idempotent if $(A^2) = A$.

Definition 3 (cf. Khan [5]) An ordered semigroup S is called semisimple if every ideal of S is idempotent.

Definition 4 A right (resp., left) ideal A of an ordered semigroup S is called prime right (resp., left) ideal of S if for all right (resp., left) ideals B, C of S , $BC \subseteq A$ implies either $B \subseteq A$ or $C \subseteq A$. A is called a semiprime if $B^2 \subseteq A$ implies $B \subseteq A$.

Obviously, every prime left (right) ideal of an ordered semigroup S is semi-prime.

Definition 5 (cf. Khan [5]) An ordered semigroup S is called left (resp., right) weakly regular if every left (resp., right) ideal of S is idempotent.

Lemma 6 (cf:[17]) An ordered semigroup $(S, \leq)$ is left (resp., right) simple if and only if for every $a \in S$, $(Sa) = S$ (resp., $(aS) = S$).

Lemma 7 (cf:[17]) . Let $(S, \leq)$ be an ordered semigroup. Then the followings assertions are equivalent.

- (1) $(x)_N$ is a left simple subsemigroup of S , for every $x \in S$.
- (2) Every left ideal of S is a right ideal of S and semiprime.

An ordered semigroup S is regular (see [8,16]) if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$.

Equivalent definitions are

- (1) $(\forall a \in S)(a \in (aSa))$,
- (2) $(\forall a \in S)(A \subseteq (aSa))$.

An ordered semigroup $(S, \leq)$ is left (resp., right) regular [16] if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$ (resp., $a \leq a^2x$).

Equivalent definitions are

- (1) $(\forall a \in S)(a \in (Sa^2))$ (resp., $a \in (a^2S)$),
- (2) $(\forall A \subseteq S)(A \subseteq (SA^2))$ (resp., $A \subseteq (A^2S)$).

An ordered semigroup $(S, \leq)$ is intraregular (see [5,16]) if for every $a \in S$, there exist $x, y \in S$ such that $a \leq xa^2y$.

Equivalent definitions are

- (1) $(\forall a \in S)(a \in (Sa^2S))$,
- (2) $(\forall A \subseteq S)(A \subseteq (SA^2S))$.

An ordered semigroup S is called completely regular (see [5,16]) if it is reg-ular, left regular, and right regular.

Lemma 8 (cf: [5]) An ordered semigroup S is completely regular if and only if $A \subseteq (A^2SA^2)$ for every $A \in S$. Equivalently, $a \in (a^2Sa^2)$ for every $a \in S$.

SOFT SETS

In [25], Sezgin and Atagun introduced some new operations on soft set theory and defined soft sets in the following way.

Let U be an initial universe set, E a set of parameters, $P(U)$ the power set of U and $A \subseteq E$. Then a soft set f_A over U is a function defined by:

$$f_A: E \rightarrow P(U) \text{ such that } f_A(x) = \emptyset, \text{ if } x \notin A.$$

Here f_A is called an approximate function. A soft set over U can be represented by the set of ordered pairs

$$f_A := \{(x, f_A(x)) : x \in E; f_A(x) \in P(U)\}.$$

It is clear that a soft set is a parameterized family of subsets of U . The set of all soft sets is denoted by $S(U)$.

Definition 9 Let $f_A, f_B \in S(U)$. Then f_A is a soft subset of f_B , denoted by $f_A \subseteq f_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in S$. Two soft sets f_A, f_B are said to be equal soft sets if $f_A \subseteq f_B$ and $f_B \subseteq f_A$ and is denoted by $f_A \cong f_B$.

Definition 10 Let $f_A, f_B \in S(U)$. Then the union of f_A and f_B , denoted by $f_A \cup f_B$, is defined by $f_A \cup f_B = f_{A \cup B}$, where $f_{A \cup B}(x) = f_A(x) \cup f_B(x)$, for all $x \in E$.

Definition 11 Let $f_A, f_B \in S(U)$. Then the intersection of f_A and f_B , denoted by $f_A \cap f_B$, is defined by $f_A \cap f_B = f_{A \cap B}$, where $f_{A \cap B}(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.

Throughout this paper, let $E = S$, where S is an ordered semigroup, unless otherwise stated.

The following definitions and results have been taken from the sources [24,25].

Definition 12 (cf. [10]) Let $f_A, f_B \in S(U)$. Then the soft product of f_A and f_B , denoted by $f_A \tilde{\circ} f_B$, is defined by

$$(f_A \tilde{\circ} f_B)(x) := \begin{cases} \bigcup_{(y,z) \in A_x} f_A(y) \cap f_B(z) & \text{if } A_x \neq \emptyset \\ \emptyset & \text{if } A_x = \emptyset \end{cases}$$

where $A_x := \{(y, z) \in S \times S / x \leq yz\}$.

Definition 2.5. (Jun et al. [10]) A double-framed soft set of A over U (briefly, DFS-set of A), where f_A^+ and f_A^- are mappings from A to $P(U)$. The set of all DFS-set of A over U will be denoted by $\text{DFS}(U)$.

For a DFS-set $\langle (f_A^+, f_A^-); A \rangle$ of A and two subsets of U , the γ -inclusive set and the δ -exclusive set of $\langle (f_A^+, f_A^-); A \rangle$, denoted by $i_A(f_A^+; \gamma)$ and $e_A(f_A^-; \delta)$, respectively, are defined as follows:

$$i_A(f_A^+; \gamma) := \{x \in A / f_A^+(x) \supseteq \gamma\}$$

$$\text{and } e_A(f_A^-; \delta) := \{x \in A / f_A^-(x) \subseteq \delta\}$$

respectively. The set

$$\text{DFA } \langle (f_A^+, f_A^-); (\gamma, \delta) \rangle := \{x \in A / f_A^+(x) \supseteq \gamma, f_A^-(x) \subseteq \delta\}$$

is called a double framed including set (cf. [10]) of $\langle (f_A^+, f_A^-); A \rangle$.

It is clear that $\text{DFA } \langle (f_A^+, f_A^-); (\gamma, \delta) \rangle := i_A(f_A^+; \gamma) \cap e_A(f_A^-; \delta)$.

Definition 13 (cf. [6]) Let $f_A = \langle (f_A^+, f_A^-); A \rangle$; A and $g_A = \langle (g_A^+, g_A^-); A \rangle$ be two double-framed soft sets of an ordered semigroup $(S, \leq)$ over U . Then the uni-int soft product, denoted by $f_A \diamond g_A = \langle (f_A^+ \tilde{\circ} g_A^+, f_A^- \tilde{*} g_A^-); A \rangle$ is defined by to be a double-framed soft set of S over U , in which $f_A^+ \tilde{\circ} g_A^+$, and $f_A^- \tilde{*} g_A^-$ are mappings from S to $P(U)$, given as follows:

$$\begin{aligned} f_A^+ \tilde{\circ} g_A^+ : S &\rightarrow P(U); x \\ \mapsto \begin{cases} \bigcup_{(y,z) \in A_x} f_A^+(y) \cap g_A^+(z) & \text{if } A_x \neq \emptyset \\ \emptyset & \text{if } A_x = \emptyset \end{cases} \end{aligned}$$

$$\begin{aligned} f_A^- \tilde{*} g_A^- : S &\rightarrow P(U); x \\ \mapsto \begin{cases} \bigcap_{(y,z) \in A_x} f_A^-(y) \cup g_A^-(z) & \text{if } A_x \neq \emptyset \\ \emptyset & \text{if } A_x = \emptyset \end{cases} \end{aligned}$$

Let $\langle (f_A^+, f_A^-); A \rangle$ and $\langle (g_B^+, g_B^-); B \rangle$ be double-framed soft sets over a common universe U . Then $\langle (f_A^+, f_A^-); A \rangle$ is called a double-framed soft subset (cf. [12]) (briefly DFS subset) of $\langle (g_B^+, g_B^-); B \rangle$, denoted by $\langle (f_A^+, f_A^-); A \rangle \subseteq \langle (g_B^+, g_B^-); B \rangle$ if

- (i) $A \subseteq B$,
- (ii) $(\forall e \in A) \begin{pmatrix} f_A^+ \text{ and } g_B^+ \text{ are identical approximations} \\ f_A^- \text{ and } g_B^- \text{ are identical approximations} \end{pmatrix}$

Two double-framed soft sets $\langle (f_A^+, f_A^-); A \rangle$ and $\langle (g_B^+, g_B^-); B \rangle$ are said to be equal, denoted by $\langle (f_A^+, f_A^-); A \rangle = \langle (g_B^+, g_B^-); B \rangle$; if $\langle (f_A^+, f_A^-); A \rangle \subseteq \langle (g_B^+, g_B^-); B \rangle$ and $\langle (g_B^+, g_B^-); B \rangle \subseteq \langle (f_A^+, f_A^-); A \rangle$

For two DFS sets $\langle (f_A^+, f_A^-); A \rangle$ and $\langle (g_A^+, g_A^-); A \rangle$ over U , the DFS int-uni set (cf. [10]) of $\langle (f_A^+, f_A^-); A \rangle$ and $\langle (g_A^+, g_A^-); A \rangle$ is defined to be a DFS set

$\langle f_A^+ \cap g_A^+, f_A^- \cup g_A^-; A \rangle$ where $f_A^+ \cap g_A^+$ and $f_A^- \cup g_A^-$ are mappings given by

$$f_A^+ \cap g_A^+ : A \rightarrow P(U), x \rightarrow f_A^+(x) \cap g_A^+(x) \\ f_A^- \cup g_A^- : A \rightarrow P(U), x \rightarrow f_A^-(x) \cup g_A^-(x).$$

It is denoted by $\langle (f_A^+, f_A^-); A \rangle \cap \langle (g_A^+, g_A^-); A \rangle = \langle f_A^+ \cap g_A^+, f_A^- \cup g_A^-; A \rangle$.

Definition 14 (cf. [6]) A double-framed soft set $\langle (f_A^+, f_A^-); A \rangle$ of S over U is called a double-framed soft semigroup (briefly, DFS semigroup) of S over U if it satisfies:

$$f_A^+(xy) \supseteq f_A^+(x) \cap f_A^+(y), f_A^-(xy) \subseteq f_A^-(x) \cup f_A^-(y), \\ \text{for all } x, y \in S.$$

Proposition 15 (cf. [6]) A DFS set $f_A = \langle (f_A^+, f_A^-); A \rangle$ of S over U is a DFS sub-semigroup of S over U if and only if $f_A \diamond f_A \subseteq f_A \vee f_A$; i.e., $f_A^+ \diamond f_A^+ \subseteq f_A^+$ and $f_A^- \diamond f_A^- \subseteq f_A^-$.

Definition 16 (cf. [6]) A double-framed soft set $\langle (f_A^+, f_A^-); A \rangle$ of S over U is called

- (i) a double-framed soft l-ideal (briefly, DFS l-ideal) of S over U if (a) $f_A^+(xy) \supseteq f_A^+(y)$ and $f_A^-(xy) \subseteq f_A^-(y)$ and (b) $x \leq y \Rightarrow f_A^+(x) \supseteq f_A^+(y)$ and $f_A^-(x) \subseteq f_A^-(y)$ for all $x, y \in S$.
- (ii) a double-framed soft r-ideal (briefly, DFS r-ideal) of S over U if (a) $f_A^+(xy) \supseteq f_A^+(x)$ and $f_A^-(xy) \subseteq f_A^-(x)$ and (b) $x \leq y \Rightarrow f_A^+(x) \supseteq f_A^+(y)$ and $f_A^-(x) \subseteq f_A^-(y)$ for all $x, y \in S$.
- (iii) a double-framed soft ideal (briefly, DFS ideal) of S over U if it is both a double-framed soft l- and r-ideal of S over U .

Definition 17 (cf. [6]) A double-framed soft set $f_A = \langle (f_A^+, f_A^-); A \rangle$ of S over U is called an idempotent if $f_A \diamond f_A = f_A$ i.e., $f_A^+ \diamond f_A^+ = f_A^+$ and $f_A^- \diamond f_A^- = f_A^-$.

Let A be a nonempty subset of S , then the characteristic double-framed soft mapping of A , denoted by $\langle (X_A^+, X_A^-); A \rangle = X_A$, is defined to be a double-framed soft set, in which A ; and A are soft mappings over U , given as follows:

$$X_A^+ : S \rightarrow P(U); x \mapsto \begin{cases} U & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases} \text{ and} \\ X_A^- : S \rightarrow P(U); x \mapsto \begin{cases} \emptyset & \text{if } x \in A \\ U & \text{if } x \notin A \end{cases}$$

Note that the characteristic mapping of the whole set S , denoted by $\langle (X_S^+, X_S^-); S \rangle$, is called the identity double-framed soft mapping, where $X_S^+(x) = U$ and $X_S^-(x) = \emptyset$, for all $x \in S$.

Lemma 18 (cf. [7]) For a nonempty subset A of S , the following assertions are equivalent:

- (1) A is a left (right or bi-ideal) of S .
- (2) The double-framed soft set $\langle (X_A^+, X_A^-); A \rangle$ of S over U is a double-framed soft l-ideal (resp., double-framed soft r- or bi-ideal) of S over U .

Lemma 19 (Khan et al. [5]) Let S be an ordered semigroup and A, B are non-empty subsets of S , then the following are equivalent:

- (i) $A \subseteq B$ if and only if $X_A \subseteq X_B$,
- (ii) $X_A \cap X_B = X_{A \cap B}$,
- (iii) $X_A \cup X_B = X_{A \cup B}$,
- (iv) $X_A \diamond X_B = X_{(AB)}$.

Theorem 20 Let S be a regular ordered semigroup. Then the following assertions are equivalent:

- (1) S is left (resp., right) simple.
- (2) Every double-framed soft l-ideal (resp., double-framed soft r-ideal) of S over U is a constant mapping.

Proof. (1) $\Rightarrow$ (2). Let S be a left simple ordered semigroup, $\langle (f_A^+, f_A^-); A \rangle$ a double-framed l-ideal of S over U , and let $a \in S$. We consider the following set:

$$E_S := \{e \in S / e^2 \geq e\}.$$

Then $E_S = \emptyset$. In fact, since S is regular and $a \in S$, there exists $x \in S$ such that $a \leq axa$. By definition it follows that

$$(ax)^2 = (axa)x \geq ax;$$

so $ax \in E_S$ and hence $E_S \neq \emptyset$.

(1) $\langle (f_A^+, f_A^-); A \rangle$ is a constant mapping on E_S . Let $t \in E_S$. Since S is left

simple and $t \in S$, we have $(St) = S$. Since $e \in S$, $e \in (St)$, so there exists $z \in S$

such that $e \leq zt$. Hence $e^2 \leq (zt)(zt) = (ztz)t$. Since $\langle (f_A^+, f_A^-); A \rangle$ is a double-

framed l-ideal of S over U , we have

$$f_A^+(e^2) \supseteq f_A^+(ztz)t \supseteq f_A^+(t) \text{ and } f_A^-(e^2) \subseteq f_A^-(ztz)t \subseteq f_A^-(t).$$

Since $e \in E_S$, we have $e^2 \geq e$ and so $f_A^+(e) \supseteq f_A^+(e^2)$, $f_A^-(e) \subseteq f_A^-(e^2)$.

Besides, since S is left simple and $e \in S$, we have $(Se) = S$. Since $t \in S = (Se)$, so there exists $y \in S$ such that $t \leq ye$. Hence $t^2 \leq (ye)(ye) = (yey)e$. Since $\langle (f_A^+, f_A^-); A \rangle$ is a double-framed l-ideal of S over U , we have

$$f_A^+(t^2) \supseteq f_A^+(yey)e \supseteq f_A^+(e) \text{ and } f_A^-(t^2) \subseteq f_A^-(yey)e \subseteq f_A^-(e).$$

On the other hand, for $t \in E_S$, we have $t^2 \geq t$ and so $f_A^+(t) \supseteq f_A^+(t^2)$ and $f_A^-(t) \subseteq f_A^-(t^2)$. Hence, $f_A^+(t) = f_A^+(e)$ and $f_A^-(t) = f_A^-(e)$.

(2) $\langle (f_A^+, f_A^-); A \rangle$ is a constant mapping on S . Let $a \in S$. Since S is regular,

there exists $x \in S$ such that $a \leq axa$. Consider the element $xa \in S$. Then it follows by definition that

$$(xa)^2 = x(axa) \geq xa;$$

Hence, $xa \in E_S$ and by part (1), we have $f_A^+(t) = f_A^+(xa)$ and $f_A^-(t) = f_A^-(xa)$. Since $\langle (f_A^+, f_A^-); A \rangle$ is a double-framed l-ideal of S over U , we have $f_A^+(t) = f_A^+(xa) \supseteq f_A^+(a)$ and $f_A^-(t) = f_A^-(xa) \subseteq f_A^-(a)$. Thus, $f_A^+(t) = f_A^+(a)$ and $f_A^-(t) = f_A^-(a)$. On the other hand, since S is left simple, and $t \in S$, $S = (St)$; we have $a \leq st$ for some $s \in S$. Since $\langle (f_A^+, f_A^-); A \rangle$ is a double framed l-ideal of S over U , we have $f_A^+(a) \supseteq f_A^+(st) \supseteq f_A^+(t) \supseteq f_A^+(a)$ and $f_A^-(a) \subseteq f_A^-(st) \subseteq f_A^-(t) \subseteq f_A^-(a)$. Thus, $f_A^+(a) = f_A^+(t)$ and $f_A^-(a) = f_A^-(t)$. Similarly, we can prove the case for the double-framed soft r-ideal of S over U .

(2) $\Rightarrow$ (1). Let $a \in S$. Then the set (Sa) is a left ideal of S . In fact, $S(Sa) \subseteq (S)(Sa) \subseteq (SSa) \subseteq (Sa)$. If $x \in (Sa)$ and $S \ni y \leq x$, then $y \in ((Sa)) = (Sa)$. Hence (Sa) is a left ideal of S and by Lemma 18, the characteristic double-framed

$$\text{soft set } X_{(Sa)} = \langle (X_{(Sa)}^+, X_{(Sa)}^-); (Sa) \rangle \text{ of } (Sa) \text{ over } U,$$

$$X_{(Sa]} : S \rightarrow P(U) \times P(U), x \mapsto \begin{cases} X_{(Sa]}^+(x) = \begin{cases} U & \text{if } x \in (Sa] \\ \emptyset & \text{if } x \notin (Sa] \end{cases} \\ X_{(Sa]}^-(x) = \begin{cases} \emptyset & \text{if } x \in (Sa] \\ U & \text{if } x \notin (Sa] \end{cases} \end{cases}$$

is a double-framed soft l-ideal of S over U . By hypothesis, $\langle (X_{(Sa]}^+, X_{(Sa]}^-); (Sa] \rangle$

is a constant mapping; that is, there exists subsets γ, δ of U such that

$$X_{(Sa]}^+(x) := \gamma \text{ and } X_{(Sa]}^-(x) := \delta.$$

Let $(Sa] \subset S$ and $t \in S$ be such that $t \notin (Sa]$. Then, $X_{(Sa]}^+(x) := \emptyset$; and $X_{(Sa]}^-(x) := U$: On the other hand, since $a^2 \in (Sa]$, we have $X_{(Sa]}^+(x) := U$ and

$X_{(Sa]}^-(x) := \emptyset$. This is a contradiction to the fact that $\langle (X_{(Sa]}^+, X_{(Sa]}^-); (Sa] \rangle$ is a constant mapping. Hence, $(Sa] = S$.

From Theorem 20, we have the following corollary:

Corollary 21 Let S be a regular ordered semigroup. Then the following assertions are equivalent:

- (1) S is simple.
- (2) Every double-framed soft ideal of S over U is a constant mapping.

Theorem 22 An ordered semigroup $(S, \leq)$ is left regular if and only if for each double-framed l-ideal $\langle (f_A^+, f_A^-); A \rangle$ of S , one has

$$f_A^+(a) = f_A^+(a^2) \text{ and } f_A^-(a) = f_A^-(a^2) \text{ for all } a \in S.$$

Proof. Suppose that $\langle (f_A^+, f_A^-); A \rangle$ is a double-framed l-ideal of S and let $a \in S$. Since S is left regular, there exists $x \in S$ such that $a \leq xa^2$. Since $\langle (f_A^+, f_A^-); A \rangle$ is a double-framed l-ideal of S , we have

$$f_A^+(a) \supseteq f_A^+(xa^2) \supseteq f_A^+(a^2) \supseteq f_A^+(a) \text{ and } f_A^-(a) \subseteq f_A^-(xa^2) \subseteq f_A^-(a^2) \subseteq f_A^-(a). \text{ Conversely, let } a \in S. \text{ We consider the left ideal } L(a^2) = (a^2 \cup Sa^2] \text{ of } S,$$

generated by a^2 . Then by Lemma 18, the characteristic double-framed soft set

$$\langle (X_{L(a^2)}^+, X_{L(a^2)}^-); L(a^2) \rangle \text{ of } L(a^2) \text{ over } U$$

$$X_{L(a^2)} : S \rightarrow P(U) \times P(U), x \mapsto \begin{cases} X_{L(a^2)}^+(x) = \begin{cases} U & \text{if } x \in L(a^2) \\ \emptyset & \text{if } x \notin L(a^2) \end{cases} \\ X_{L(a^2)}^-(x) = \begin{cases} \emptyset & \text{if } x \in L(a^2) \\ U & \text{if } x \notin L(a^2) \end{cases} \end{cases}$$

is a double-framed soft l-ideal of S over U . By hypothesis, we have $X_{L(a^2)}^+(a) = X_{L(a^2)}^+(a^2)$ and $X_{L(a^2)}^-(a) = X_{L(a^2)}^-(a^2)$. Since $a^2 \in L(a^2)$, we have $X_{L(a^2)}^+(a^2) = U$ and $X_{L(a^2)}^-(a^2) = \emptyset$, and hence $X_{L(a^2)}^+(a) = U$ and $X_{L(a^2)}^-(a) = \emptyset$; Thus, $a \in L(a^2) = (a^2 \cup Sa^2]$ and $a \leq y$, for some $y \in a^2 \cup Sa^2$: If $y = a^2$ then $a \leq y = a^2 = aa$ and $a \in Sa^2$ and $a \in (Sa^2]$. If $y = xa^2$ for some $x \in S$, then $a \leq y = xa^2 \in Sa^2$, and $a \in Sa^2$.

From left-right dual of Theorem 22, we have the following theorem.

Theorem 23 An ordered semigroup $(S, \leq)$ is right regular if and only if for each double-framed r-ideal $\langle (f_A^+, f_A^-); A \rangle$ of S , one has

$$f_A^+(a) = f_A^+(a^2) \text{ and } f_A^-(a) = f_A^-(a^2) \text{ for all } a \in S.$$

From ([6], Theorem 8) and by Theorems 22, 23, and Lemma 8, we have the following characterization theorem for completely regular ordered semigroups.

Theorem 24 Let $(S, \leq)$ be an ordered semigroup. Then the following state-ments are equivalent.

- (1) S is completely regular.
- (2) For each double-framed soft bi-ideal $\langle (f_A^+, f_A^-); A \rangle$ of S over U , one has

$$f_A^+(a) = f_A^+(a^2) \text{ and } f_A^-(a) = f_A^-(a^2) \text{ for all } a \in S.$$

- (3) For each double-framed soft l-ideal $\langle (f_A^+, f_A^-); A \rangle$ and each double-framed soft r-ideal $\langle (g_A^+, g_A^-); A \rangle$ of S over U , one has

$$f_A^+(a) = f_A^+(a^2) \text{ and } f_A^-(a) = f_A^-(a^2)$$

$$g_A^+(a) = g_A^+(a^2) \text{ and } g_A^-(a) = g_A^-(a^2) \text{ for all } a \in S.$$

Mainly, Kehayopulu studied the ideal theory of ordered semigroups (see [8]). She extended the notion of ideal into fuzzy context and defined the notions of fuzzy left (resp., right) ideals, fuzzy bi-ideal and so on for ordered semigroups [17,18]. In the following we extend the notion of prime ideals of ordered semi-groups into double-framed soft sets given by Jun et al. in [10].

PRIME DOUBLE-FRAMED SOFT IDEALS

In this section, we define prime and semiprime double-framed soft ideals and

give the main results for these notions.

Lemma 25 Let $A = \langle (f_A^+, f_A^-); A \rangle$, $g_B = \langle (g_B^+, g_B^-); B \rangle$ and $h_C = \langle (h_C^+, h_C^-); C \rangle$ be any three DFS sets of S over U . Then

$$(1) f_A \sqcup (g_B \sqcap h_C) = (f_A \sqcup g_B) \sqcap (f_A \sqcup h_C); (2) f_A \sqcap (g_B \sqcup h_C) = (f_A \sqcap g_B) \sqcup (f_A \sqcap h_C);$$

Proof. Straightforward.

Let $DFS(U)$ denote the set of all double-framed soft sets of S over U , unless otherwise stated.

Proposition 26 Let $f_A, g, h_C \in DFS(U)$: Then we have,

- (i) $f_A \diamond (g_B \diamond h_C) = (f_A \diamond g_B) \diamond h_C$;
- (ii) $f_A \diamond g_B \neq g_B \diamond f_A$; in general. However, if S is commutative, then $f_A \diamond g_B = g_B \diamond f_A$,
- (iii) $f_A \diamond (g_B \tilde{\cup} h_C) = (f_A \diamond g_B) \tilde{\cup} (f_A \diamond h_C)$, and $(g_B \tilde{\cup} h_C) \diamond f_A = (g_B \tilde{\cup} f_A) \diamond (h_C \tilde{\cup} f_A)$.
- (iv) $f_A \diamond (g_B \tilde{\cap} h_C) = (f_A \diamond g_B) \tilde{\cap} (f_A \diamond h_C)$, and $(g_B \tilde{\cap} h_C) \diamond f_A = (g_B \tilde{\cap} f_A) \diamond (h_C \tilde{\cap} f_A)$.
- (v) If $t_D, l_F \in DFS(U)$ such that $t_D \sqsubseteq f_A$ and $l_F \sqsubseteq g_B$; then $t_D \diamond l_F \sqsubseteq f_A \diamond g_B$:

Proof. (i) and (ii) are straightforward.

(iii) Let $x \in S$. If $A_x = \emptyset$, then

$$(f_A^+ \tilde{\cup} (g_B^+ \tilde{\cup} h_C^+))(x) = \emptyset = (f_A^+ \tilde{\cup} g_B^+)(x) \tilde{\cup} (f_A^+ \tilde{\cup} h_C^+)(x)$$

$$= ((f_A^+ \tilde{\cup} g_B^+) \tilde{\cup} (f_A^+ \tilde{\cup} h_C^+))(x)$$

$$(f_A^+ \tilde{\cap} (g_B^+ \tilde{\cap} h_C^+))(x) = \emptyset$$

$$= (f_A^- \tilde{\cap} g_B^-)(x) \tilde{\cap} (f_A^- \tilde{\cap} h_C^-)(x)$$

$$((f_A^- \tilde{\cap} g_B^-) \tilde{\cap} (f_A^- \tilde{\cap} h_C^-))(x).$$

$$\text{Let } A_x \neq \emptyset \text{ then } (f_A^+ \tilde{\cup} (g_B^+ \tilde{\cup} h_C^+))(x)$$

$$= \bigcup_{(y,z) \in A_x} \{ (f_A^+(y) \cap (g_B^+(z) \cup h_C^+(z))) \}$$

$$= \bigcup_{(y,z) \in A_x} \{ (f_A^+(y) \cap (g_B^+(z) \cup h_C^+(z))) \}$$

$$= \bigcup_{(y,z) \in A_x} \{ (f_A^+(y) \cap g_B^+(z)) \cup (f_A^+(y) \cap h_C^+(z)) \}$$

$$= \bigcup_{(y,z) \in A_x} (f_A^+(y) \cap g_B^+(z)) \cup \bigcup_{(y,z) \in A_x} \{ (f_A^+(y) \cap h_C^+(z)) \}$$

$$\begin{aligned}
&= (f_A^+ \tilde{o} g_B^+)(x) \cup (f_A^+ \tilde{o} h_c^+)(x) \\
&= ((f_A^+ \tilde{o} g_B^+) \tilde{\cup} (f_A^+ \tilde{o} h_c^+))(x)
\end{aligned}$$

and

$$\begin{aligned}
(f_A^- \tilde{*} (g_B^- \tilde{\cap} h_c^-))(x) &= \\
&\bigcap_{(y,z) \in A_x} \{(f_A^-(y) \cup (g_B^-(z) \cap h_c^-(z)))\} \\
&= \bigcap_{(y,z) \in A_x} \{(f_A^-(y) \cup (g_B^-(z) \cap h_c^-(z)))\} \\
&= \bigcap_{(y,z) \in A_x} \{(f_A^-(y) \cup g_B^-(z)) \cap (f_A^-(y) \cup h_c^-(z))\} \\
&= \bigcap_{(y,z) \in A_x} \{(f_A^-(y) \cup g_B^-(z)) \cap \bigcap_{(y,z) \in A_x} \{(f_A^-(y) \cup h_c^-(z))\} \\
&= (f_A^- \tilde{*} g_B^-)(x) \cap \\
(f_A^- \tilde{*} h_c^-)(x) &= ((f_A^- \tilde{*} g_B^-) \tilde{\cap} (f_A^- \tilde{*} h_c^-))(x).
\end{aligned}$$

Similarly, we can prove that $(g_B^+ \tilde{\cup} h_c^+) \tilde{o} f_A^+ \cong (g_B^+ \tilde{o} f_A^+) \tilde{\cup} (h_c^+ \tilde{o} f_A^+)$ and $(g_B^- \tilde{\cap} h_c^-) \tilde{*} f_A^- \cong (g_B^- \tilde{*} f_A^-) \tilde{\cap} (h_c^- \tilde{*} f_A^-)$.

(iv) If $A_x = \emptyset$, then the result is obviously true. Let $A_x \neq \emptyset$, then

$$\begin{aligned}
(f_A^+ \tilde{o} (g_B^+ \tilde{\cap} h_c^+))(x) &= \bigcup_{(y,z) \in A_x} \{(f_A^+(y) \cap (g_B^+(z) \cap h_c^+(z)))\} \\
&= \bigcup_{(y,z) \in A_x} \{(f_A^+(y) \cap (g_B^+(z) \cap h_c^+(z)))\}
\end{aligned}$$

$$\begin{aligned}
&= \\
&= \bigcup_{(y,z) \in A_x} \{(f_A^+(y) \cap g_B^+(z)) \cap (f_A^+(y) \cap h_c^+(z))\} \\
&= \bigcup_{(y,z) \in A_x} (f_A^+(y) \cap g_B^+(z) \cap \bigcup_{(y,z) \in A_x} \{(f_A^+(y) \cap h_c^+(z))\} \\
&= (f_A^+ \tilde{o} g_B^+)(x) \cap (f_A^+ \tilde{o} h_c^+)(x) \\
&= ((f_A^+ \tilde{o} g_B^+) \tilde{\cap} (f_A^+ \tilde{o} h_c^+))(x)
\end{aligned}$$

and

$$\begin{aligned}
(f_A^- \tilde{*} (g_B^- \tilde{\cup} h_c^-))(x) &= \\
&\bigcap_{(y,z) \in A_x} \{(f_A^-(y) \cup (g_B^-(z) \cup h_c^-(z)))\} \\
&= \bigcap_{(y,z) \in A_x} \{(f_A^-(y) \cup (g_B^-(z) \cup h_c^-(z)))\} \\
&= \bigcap_{(y,z) \in A_x} \{(f_A^-(y) \cup g_B^-(z)) \cup (f_A^-(y) \cup h_c^-(z))\}
\end{aligned}$$

Similarly, we can prove the second part.

(v) Follows from ([6], Proposition 9 (i)).

Theorem 27 A nonempty subset A of an ordered semigroup S is a subsemi-group of S if and only if the double-framed soft set $\langle (g_B^+, g_B^-); B \rangle$, defined by

$$\begin{aligned}
g_B^+ : S \rightarrow P(U); x \mapsto &\begin{cases} \gamma_1 & \text{if } x \in A \\ \gamma_2 & \text{if } x \in S \cap A \end{cases} \\
g_B^- : S \rightarrow P(U); x \mapsto &\begin{cases} \delta_1 & \text{if } x \in A \\ \delta_2 & \text{if } x \in S \cap A \end{cases}
\end{aligned}$$

is a double-framed soft semigroup of S over U , where $\gamma_1, \gamma_2, \delta_1, \delta_2 \subseteq U$ such that $\gamma_2 \subseteq \gamma_1$, and $\delta_1 \subseteq \delta_2$.

Proof. Suppose that A is a subsemigroup of S . Let $x, y \in S$. We discuss the

following two cases. 1. If $x, y \in A$. Then $xy \in A$ and we have

$$\begin{aligned}
g_B^+(xy) &= \gamma_1 = g_B^+(y) \text{ and } g_B^-(xy) = \delta_1 = g_B^-(y). \\
\text{Hence } g_B^+(xy) &= \gamma_1 = \gamma_1 \cap \gamma_1 = g_B^+(x) \cap g_B^+(y) \text{ and } g_B^-(xy) = \delta_1 \\
&= \delta_1 \cap \delta_1 = g_B^-(x) \cup g_B^-(y).
\end{aligned}$$

Case 2. If one of x or y does not belong to A , then $g_B^+(x) \cap g_B^+(y) = \gamma_2 \subseteq g_B^+(xy)$ and $g_B^-(x) \cup g_B^-(y) = \delta_2 \supseteq g_B^-(xy)$. Therefore $\langle (g_B^+, g_B^-); B \rangle$, is a DFS semigroup of S over U .

Conversely, assume that $\langle (g_B^+, g_B^-); B \rangle$, is a DFS semigroup of S over U .

Let $x, y \in A$. Then

$$g_B^+(x) = \gamma_1 = g_B^+(y) \text{ and } g_B^-(x) = \delta_1 = g_B^-(y).$$

By hypothesis,

$$g_B^+(xy) \supseteq g_B^+(x) \cap g_B^+(y) = \gamma_1 \text{ and } g_B^-(xy) \subseteq g_B^-(x) \cup g_B^-(y) = \delta_1.$$

That is, $g_B^+(xy) \supseteq \gamma_1$ and $g_B^-(xy) \subseteq \delta_1$ and so $xy \in A$. Therefore, A is a subsemigroup of S .

Theorem 28 A nonempty subset A of an ordered semigroup S is a left (resp., right) ideal of S if and only if the double-framed soft set $\langle (g_B^+, g_B^-); B \rangle$, defined by

$$\begin{aligned}
g_B^+ : S \rightarrow P(U); x \mapsto &\begin{cases} \gamma_1 & \text{if } x \in A \\ \gamma_2 & \text{if } x \in S \cap A \end{cases} \\
g_B^- : S \rightarrow P(U); x \mapsto &\begin{cases} \delta_1 & \text{if } x \in A \\ \delta_2 & \text{if } x \in S \cap A \end{cases}
\end{aligned}$$

is a double-framed soft left (resp., right) -ideal of S over U , where $\gamma_1, \gamma_2, \delta_1, \delta_2 \subseteq U$ such that $\gamma_2 \subseteq \gamma_1$, and $\delta_1 \subseteq \delta_2$.

Proof. Suppose that A is a subsemigroup of S . Let $x, y \in S$ be such that $x \cdot y$. If $y \notin A$, then $g_B^+(y) = \gamma_2 \subseteq g_B^+(x)$ and $g_B^-(y) = \delta_1 \supseteq g_B^-(x)$. If $y \in A$, then $\gamma_1 = g_B^+(y)$ and $\delta_1 = g_B^-(y)$. Since $x \leq y \in A$, and A is a left ideal of S , we have $x \in A$. Then

$$g_B^+(x) = \gamma_1 = g_B^+(y) \text{ and } g_B^-(x) = \delta_1 = g_B^-(y).$$

For $x, y \in S$, we discuss the following two cases.

Case 1. If $x \in S$; and $y \in A$. Then $xy \in A$ and we have

$$\begin{aligned}
g_B^+(xy) &= \gamma_1 = g_B^+(y) \text{ and } g_B^-(xy) = \delta_1 = g_B^-(y). \\
\text{Case 2. If } x \in S \text{ and } y \notin A, &\text{ then } g_B^+(y) = \gamma_2 \subseteq g_B^+(xy) \text{ and } g_B^-(y) = \delta_1 \\
&= \delta_2 \supseteq g_B^-(xy). \text{ Therefore } \langle (g_B^+, g_B^-); B \rangle, \text{ is a DFS l-ideal of } S \text{ over } U.
\end{aligned}$$

Similarly, we can prove the result for a DFS r-ideal of S over U .

Conversely, assume that $\langle (g_B^+, g_B^-); B \rangle$, is a DFS l-ideal of S over U . Let $x, y \in S$ be such that $x \leq y$. If $y \in A$, then $g_B^+(y) \supseteq \gamma_1$ and $g_B^-(y) \subseteq \delta_1$. Since $g_B^+(x) \supseteq g_B^+(y) \supseteq \gamma_1$ and $g_B^-(x) \subseteq g_B^-(y) \subseteq \delta_1$, we have $x \in A$.

Let $x \in S$ and $y \in A$, then $g_B^+(y) = \gamma_1$ and $g_B^-(y) = \delta_1$. By hypothesis

$$g_B^+(xy) \supseteq g_B^+(y) = \gamma_1 \text{ and } g_B^-(xy) \subseteq g_B^-(y) = \delta_1.$$

Hence, $xy \in A$. Thus, A is a left ideal of S . Similarly, we can show that A is a right ideal of S .

Corollary 29 A nonempty subset A of an ordered semigroup S is an ideal of S if and only if the double-framed soft set $\langle (g_B^+, g_B^-); B \rangle$, defined by

$$g_B^+ : S \rightarrow P(U); x \mapsto \begin{cases} \gamma_1 & \text{if } x \in A \\ \gamma_2 & \text{if } x \in S \cap A \end{cases}$$

$$g_B^- : S \rightarrow P(U); x \mapsto \begin{cases} \delta_1 & \text{if } x \in A \\ \delta_2 & \text{if } x \in S \cap A \end{cases}$$

is a double-framed soft ideal of S over U , where $\gamma_1, \gamma_2, \delta_1, \delta_2 \subseteq U$ such that $\gamma_2 \subseteq \gamma_1$, and $\delta_1 \subseteq \delta_2$.

Proof. Follows from Theorem 28.

PRIME DOUBLE-FRAMED SOFT IDEALS

Definition 30 A double-framed soft l-ideal (r-ideal) $f_A = \langle (f_A^+, f_A^-); A \rangle$ of S over U is called a prime double-framed soft l-ideal (r-ideal) (briefly, PDFS l-ideal (r-ideal)) of S over U , if for any double-framed soft l-ideals (r-ideals) $g_B = \langle (g_B^+, g_B^-); B \rangle$ and $h_C = \langle (h_C^+, h_C^-); C \rangle$ of S over U such that $g_B \diamond h_C \subseteq f_A$ implies $g_B \subseteq f_A$ or $h_C \subseteq f_A$. That is, $g_B^+ \tilde{\circ} h_C^+ \subseteq f_A^+$ and $g_B^- \tilde{*} h_C^- \subseteq f_A^-$ imply $g_B^+ \subseteq f_A^+$ or $h_C^+ \subseteq f_A^+$ and $g_B^- \subseteq f_A^-$ or $h_C^- \subseteq f_A^-$.

Definition 31 A double-framed soft l-ideal (r-ideal) $f_A = \langle (f_A^+, f_A^-); A \rangle$ of S over U is called a semiprime double-framed soft l-ideal (r-ideal) (briefly, SPDFS l-ideal (r-ideal)) of S over U , if for any double-framed soft l-ideal (r-ideal) $g_B = \langle (g_B^+, g_B^-); B \rangle$ of S over U such that $g_B \diamond h_C \subseteq f_A$ implies $g_B \subseteq f_A$. That is, $g_B^+ \tilde{\circ} h_C^+ \subseteq f_A^+$ and $g_B^- \tilde{*} h_C^- \subseteq f_A^-$ imply $g_B^+ \subseteq f_A^+$ and $g_B^- \subseteq f_A^-$.

Proposition 32 Let $f_{A_i} = \langle (f_{A_i}^+, f_{A_i}^-); A_i \rangle$ be a family of prime double-framed soft ideals of S over U . Then $\bigcap_{i \in I} f_{A_i} = \langle (\bigcap_{i \in I} f_{A_i}^+, \bigcup_{i \in I} f_{A_i}^-); A_i \rangle$ is a semiprime double-framed soft ideal of S over U .

Proof. Straightforward.

Definition 33 A DFS ideal $f_A = \langle (f_A^+, f_A^-); A \rangle$ of S over U is called idempo-tent if $f_A \diamond f_A = f_A$. That is, $f_A^+ \tilde{\circ} f_A^+ = f_A^+$ and $f_A^- \tilde{*} f_A^- = f_A^-$.

Definition 34 An ordered semigroup S is called double-framed soft semisimple if every DFS ideal of S over U is idempotent.

Theorem 35 (Khan et al. [6]) Let $X_S = \langle (X_S^-, X_S^-); S \rangle$ be the identity double-framed soft set over U and $g_S = \langle (g_S^+, g_S^-); S \rangle$ be a double-framed soft set of S over U . Then $\langle (g_S^+, g_S^-); S \rangle$ is a double-framed soft l-ideal of S over U if

- (1) $(\forall x, y \in S) \ x \leq y \Rightarrow g_S^+(x) \supseteq g_S^+(y); \ g_S^-(x) \subseteq g_S^-(y)$,
- (2) $X_S \diamond g_S \subseteq g_S$.

Theorem 36 (Khan et al. [6]) Let $X_S = \langle (X_S^-, X_S^-); S \rangle$ be the identity double-framed soft set over U and $g_S = \langle (g_S^+, g_S^-); S \rangle$ be a double-framed soft set of S over U . Then the following assertion are equivalent:

- (1) $\langle (g_S^+, g_S^-); S \rangle$ is a double-framed soft r-ideal of S over U ,
- (2) $\langle (g_S^+, g_S^-); S \rangle$ satisfies the condition (DFS1) and $g_S \diamond X_S \subseteq g_S$.

Corollary 37 (Khan et al.) Let $X_S = \langle (X_S^-, X_S^-); S \rangle$ be the identity double-framed soft set over U and $g_S = \langle (g_S^+, g_S^-); S \rangle$ be a double-framed soft set of S over U . Then the following assertion are equivalent:

- (1) $\langle (g_S^+, g_S^-); S \rangle$ is a double-framed soft ideal of S over U ,
- (2) $\langle (g_S^+, g_S^-); S \rangle$ satisfies the condition (DFS1) and $X_S \diamond g_S \subseteq g_S, \ g_S \diamond X_S \subseteq g_S$.

Lemma 38 If $f_A = \langle (f_A^+, f_A^-); A \rangle$ and $g_B = \langle (g_B^+, g_B^-); B \rangle$ are DFS l-ideal and DFS r-ideal of S over U , respectively. Then $f_A \diamond g_B$ is a DFS ideal of S over U .

Proof. Let $f_A = \langle (f_A^+, f_A^-); A \rangle$ be a DFS l-ideal and $g_B = \langle (g_B^+, g_B^-); B \rangle$ be a DFS r-ideal of S over U , respectively. Then

$$X_S^+ \tilde{\circ} (f_A^+ \tilde{\circ} g_B^+) = (X_S^+ \tilde{\circ} f_A^+) \tilde{\circ} g_B^+ \subseteq f_A^+ \tilde{\circ} g_B^+$$

and

$$X_S^- \tilde{*} (f_A^- \tilde{*} g_B^-) = (X_S^- \tilde{*} f_A^-) \tilde{*} g_B^- \supseteq f_A^- \tilde{*} g_B^-$$

Also,

$$(f_A^+ \tilde{\circ} g_B^+) \tilde{\circ} X_S^+ = f_A^+ \tilde{\circ} (g_B^+ \tilde{\circ} X_S^+) \subseteq f_A^+ \tilde{\circ} g_B^+$$

and

$$(f_A^- \tilde{*} g_B^-) \tilde{*} X_S^- = f_A^- \tilde{*} (g_B^- \tilde{*} X_S^-) \supseteq f_A^- \tilde{*} g_B^-$$

Let $x, y \in S$ be such that $x \leq y$. If $(p, q) \in A_y$, then $pq \leq y \Rightarrow pq \leq x \Rightarrow (p, q) \in A_x$. Hence, $A_y \subseteq A_x$. If $A_x = \emptyset$, then $A_y = \emptyset$, and hence,

$$(f_A^+ \tilde{\circ} g_B^+)(x) = \emptyset = (f_A^+ \tilde{\circ} g_B^+)(y) \text{ and } (f_A^- \tilde{*} g_B^-)(x) = U = (f_A^- \tilde{*} g_B^-)(y). \text{ If } A_x \neq \emptyset, \text{ then } A_y \neq \emptyset, \text{ and so}$$

$$(f_A^+ \tilde{\circ} g_B^+)(y) = \bigcup_{(p,q) \in A_x} \{f_A^+(p) \tilde{\circ} g_B^+(q)\}$$

$$\subseteq \bigcup_{(c,d) \in A_x} \{f_A^+(c) \tilde{\circ} g_B^+(d)\} = (f_A^+ \tilde{\circ} g_B^+)(x)$$

and

$$(f_A^- \tilde{*} g_B^-)(y) = \bigcup_{(p,q) \in A_x} \{f_A^-(p) \tilde{*} g_B^-(q)\}$$

$$\supseteq \bigcup_{(c,d) \in A_x} \{f_A^-(c) \tilde{*} g_B^-(d)\} = (f_A^- \tilde{*} g_B^-)(x)$$

Therefore, $f_A \diamond g_B$ is a double-framed soft ideal of S over U .

Proposition 39 (Khan et al. [5]) If $f_A = \langle (f_A^+, f_A^-); A \rangle$ and $g_B = \langle (g_B^+, g_B^-); B \rangle$ are DFS r-ideal and DFS l-ideal of S over U , respectively. Then $f_A \diamond g_B \subseteq f_A \cap g_B$.

Lemma 40 If $f_A = \langle (f_A^+, f_A^-); A \rangle$ is a DFS ideal of S over U , then $f_A \sqcup (X_S \diamond f_A)$ and $f_A \sqcup (X_S \diamond f_A)$ are DFS ideal of S over U .

Proof. We have

$$X_S^+ \tilde{\circ} (f_A^+ \tilde{\circ} (X_S^+ \tilde{\circ} f_A^+)) = (X_S^+ \tilde{\circ} f_A^+) \tilde{\circ} (X_S^+ \tilde{\circ} (X_S^+ \tilde{\circ} f_A^+))$$

$$= (X_S^+ \tilde{\circ} f_A^+) \tilde{\circ} (X_S^+ \tilde{\circ} X_S^+) \tilde{\circ} f_A^+$$

$$\subseteq f_A^+ \tilde{\circ} (X_S^+ \tilde{\circ} f_A^+)$$

$$X_S^- \tilde{*} (f_A^- \tilde{*} (X_S^- \tilde{*} f_A^-)) = (X_S^- \tilde{*} f_A^-) \tilde{*} (X_S^- \tilde{*} (X_S^- \tilde{*} f_A^-))$$

$$= (X_S^- \tilde{*} f_A^-) \tilde{*} (X_S^- \tilde{*} X_S^-) \tilde{*} f_A^-$$

$$\supseteq f_A^- \tilde{*} (X_S^- \tilde{*} f_A^-)$$

and

$$(f_A^+ \tilde{\circ} (X_S^+ \tilde{\circ} f_A^+)) \tilde{\circ} X_S^+ = (f_A^+ \tilde{\circ} X_S^+) \tilde{\circ} ((X_S^+ \tilde{\circ} f_A^+) \tilde{\circ} X_S^+)$$

$$\subseteq (f_A^+ \tilde{\circ} X_S^+) \tilde{\circ} (X_S^+ \tilde{\circ} (f_A^+ \tilde{\circ} X_S^+))$$

$$(f_A^+ \tilde{\circ} (X_S^+ \tilde{\circ} f_A^+))$$

$$(f_A^- \tilde{*} (X_S^- \tilde{*} f_A^-)) \tilde{*} X_S^- = (f_A^- \tilde{*} X_S^-) \tilde{*} ((X_S^- \tilde{*} f_A^-) \tilde{*} X_S^-)$$

$$= (f_A^- \tilde{*} X_S^-) \tilde{\cap} (X_S^- \tilde{*} (f_A^- \tilde{*} X_S^-))$$

$$\supseteq (f_A^- \tilde{\cap} (X_S^- \tilde{*} f_A^-))$$

Let $x, y \in S$ be such that $x \leq y$. If $(p, q) \in A_y$, then $pq \geq y \Rightarrow pq \geq x \Rightarrow (p, q) \in A_x$.

Hence, $A_y \subseteq A_x$. If $A_x = \emptyset$, then $A_y = \emptyset$, and hence, $(f_A^+ \tilde{\cup} (X_S^+ \tilde{\delta} f_A^+))(x) = \emptyset = (f_A^+ \tilde{\cup} (X_S^+ \tilde{\delta} f_A^+))(y)$

and $(f_A^- \tilde{\cap} (X_S^- \tilde{*} f_A^-))(x) = U = (f_A^- \tilde{\cap} (X_S^- \tilde{*} f_A^-))(y)$. If $A_x \neq \emptyset$, then $A_y \neq \emptyset$ and so

$$(f_A^+ \tilde{\cup} (X_S^+ \tilde{\delta} f_A^+))(y) = (f_A^+(y) \tilde{\cup} (X_S^+ \tilde{\delta} f_A^+)(y)) \\ = (f_A^+(y) \cup \bigcup_{(p,q) \in A_y} \{X_S^+(p) \cap f_A^+(q)\})$$

$$\subseteq f_A^+(y) \cup \bigcup_{(c,d) \in A_x} \{X_S^+(c) \cap f_A^+(d)\}$$

$$(\text{Since } x \leq y \Rightarrow f_A^+(x) \supseteq f_A^+(y))$$

$$= f_A^+(x) \cup (X_S^+ \tilde{\delta} g_B^+)(x)$$

$$= (f_A^+ \cup (X_S^+ \tilde{\delta} g_B^+))(x)$$

and

$$(f_A^- \tilde{\cap} (X_S^- \tilde{*} f_A^-))(y) = (f_A^-(y) \cap (X_S^- \tilde{\delta} f_A^-))(y) \\ = (f_A^-(y) \cap \bigcap_{(p,q) \in A_y} \{X_S^-(p) \cup f_B^-(q)\})$$

$$\supseteq f_A^-(y) \cap \bigcap_{(c,d) \in x} \{X_S^-(c) \cup f_B^-(d)\}$$

$$(\text{Since } x \leq y \Rightarrow f_A^-(x) \subseteq f_A^-(y))$$

$$= f_A^-(x) \tilde{\cup} (X_S^- \tilde{*} g_B^-)(x)$$

$$= (f_A^- \tilde{\cup} (X_S^- \tilde{*} g_B^-))(x)$$

Therefore, $f_A \sqcup (X_S \diamond f_A)$ is a DFS ideal of S over U . Similarly, we can prove that $f_A \sqcup (f_A \diamond X_S)$ is a DFS ideal of S over U .

Theorem 41 Every DFS ideal of an ordered semigroup S over U is semiprime DFS ideal of S over U if and only if S is DFS semisimple.

Proof. Suppose that S is semisimple ordered semigroup and $f_A = \langle (f_A^+, f_A^-); A \rangle$ a DFS ideal of S over U . If $g_B = \langle (g_B^+, g_B^-); B \rangle$ is a DFS ideal of S over U such that $g_B^+ \tilde{\delta} g_B^+ \subseteq f_A^+$ and $g_B^- \tilde{\delta} g_B^- \subseteq f_A^-$. Then by hypothesis, $g_B^+ \subseteq f_A^+$ and $g_B^- \subseteq f_A^-$. Therefore $f_A = \langle (f_A^+, f_A^-); A \rangle$ is semiprime. Conversely, assume that every DFS ideal

of S over U is a semiprime DFS ideal of S over U . Let $g_B = \langle (g_B^+, g_B^-); B \rangle$ be a DFS ideal, then $g_B \diamond g_B$ is a DFS ideal of S over U (Lemma 38). Since $g_B \diamond g_B \subseteq g_B \diamond g_B$, so by hypothesis, $g_B \subseteq g_B \diamond g_B$. But $g_B \diamond g_B \subseteq g_B$ (Corollary 37). Thus, $g_B \diamond g_B = g_B$ and S is DFS semisimple.

Theorem 42 Every DFS ideal of an ordered semigroup S over U is a prime DFS ideal if and only if S is DFS semisimple and the set of all DFS ideals over U is totally ordered under inclusion.

Proof. Suppose that every DFS ideal of S over U is prime DFS ideal. Then every DFS ideal of S over U is semiprime. Thus, by Theorem 41, S is DFS semisimple. Now suppose that $f_A =$

$\langle (f_A^+, f_A^-); A \rangle$ and $g_B = \langle (g_B^+, g_B^-); B \rangle$ are DFS ideals of S over U . Then by Proposition 39, $f_B \diamond g_B \subseteq f_A \cap g_B$ and $f_A \cap g_B$ is a DFS ideal of S over U . By hypothesis, $f_A \subseteq f_A \cap g_B$ or $g_B \subseteq f_A \cap g_B$. This implies that either $f_A \subseteq g_B$ or $g_B \subseteq f_A$.

Conversely, assume that S is semisimple and the set of all DFS ideals of S over U is totally ordered under inclusion. Let $f_A = \langle (f_A^+, f_A^-); A \rangle$, $g_B = \langle (g_B^+, g_B^-); B \rangle$ and $h_C = \langle (h_C^+, h_C^-); C \rangle$ be DFS ideals of S over U such that $f_B \diamond g_B \subseteq h_C$. Since the set of DFS ideals of S over U is totally ordered under inclusion, so either $g_B \subseteq h_C$ or $h_C \subseteq g_B$. Assume that $g_B \subseteq h_C$. Since $g_B = g_B \diamond g_B \subseteq g_B \diamond h_C \subseteq f_A$. Thus, $f_A = \langle (f_A^+, f_A^-); A \rangle$ is prime DFS ideal of S over U .

Lemma 43 (Khan et al. [5]) Let S be an ordered semigroup with identity. Then the following conditions are equivalent.

- (1) S is semisimple.
- (2) $A \cap B = (AB]$, for all ideals A and B of S .
- (3) $A = (A^2]$, for every ideal A of S .
- (4) $A(a) = (A(a)^2]$ for every $a \in S$.

Recall that an ordered semigroup S is semisimple if and only if for each $a \in S$ there exist $x, y, z \in S$ such that $a = xayaz$ (cf. [5]).

Theorem 44 An ordered semigroup S is semisimple if and only if for all DFS ideals $f_A = \langle (f_A^+, f_A^-); A \rangle$ and $g_B = \langle (g_B^+, g_B^-); B \rangle$ of S over U , we have $f_A \diamond g_B = f_A \cap g_B$.

Proof. Suppose that S is semisimple. Let $f_A = \langle (f_A^+, f_A^-); A \rangle$ and $g_B = \langle (g_B^+, g_B^-); B \rangle$ be DFS ideals of S over U and $a \in S$. Then there exist $x, y, z \in S$

such that $a \leq xayaz = (xay)(az)$. Then $(xay, az) \in A_a$ and hence

$$(f_A^+ \tilde{\delta} g_B^+)(a) = \bigcup_{(y,z) \in A_a} \{f_A^+(y) \cap g_B^+(z)\} \\ \supseteq f_A^+(xay) \cap g_B^+(az)$$

$$(f_A^- \tilde{*} g_B^-)(a) = \bigcap_{(y,z) \in x} \{X_S^-(y) \cup f_B^-(z)\} \\ \subseteq f_A^-(xay) \cup g_B^-(az).$$

Since

$f_A = \langle (f_A^+, f_A^-); A \rangle$ and $g_B = \langle (g_B^+, g_B^-); B \rangle$ are DFS ideals of S over U , we have

$$f_A^+(xay) \supseteq f_A^+(xa) \supseteq f_A^+(a), \quad g_B^+(az) \supseteq g_B^+(a),$$

Thus, we have $(f_A^+ \tilde{\delta} g_B^+)(a) \supseteq f_A^+(a) \tilde{\cap} g_B^+(a)$ and $(f_A^- \tilde{*} g_B^-)(a) \subseteq f_A^-(a) \tilde{\cup} g_B^-(a)$. By Proposition 39, we have $f_A^+ \tilde{\cap} g_B^+ \supseteq f_A^+ \tilde{\delta} g_B^+$ and $f_A^- \tilde{\cup} g_B^- \subseteq f_A^- \tilde{*} g_B^-$. Therefore, $f_A^+ \tilde{\cap} g_B^+ \supseteq f_A^+ \tilde{\delta} g_B^+$ and $f_A^- \tilde{\cup} g_B^- \subseteq f_A^- \tilde{*} g_B^-$.

Conversely, assume that $f_A^+ \diamond g_B^+ = f_A^+ \cap g_B^+$ for every DFS ideal $f_A = \langle (f_A^+, f_A^-); A \rangle$ and $g_B = \langle (g_B^+, g_B^-); B \rangle$ of S over U . We show that S

is semisimple. By Lemma 43, it is enough to prove that

$$A \cap B = (AB], \text{ for all ideals } A \text{ and } B \text{ of } S.$$

Since A and B are ideals of S . By Lemma 18, the DFS characteristic mappings X_A and X_B of A and B respectively are DFS ideals of S over U . By hypothesis,

$$X_A \cap X_B = X_{AB}.$$

By Lemma 19, we have $X_A \cap X_B = X_{A \cap B}$ and $X_A \diamond X_B = X_{(AB]}$. Thus, we have $X_{A \cap B} = X_{(AB]}$. Again by Lemma 19 (i), we have $A \cap B = (AB]$ and S is semisimple (Lemma 43).

Corollary 45 An ordered semigroup S is semisimple if and only if for every DFS ideal $f_A = \langle (f_A^+, f_A^-); A \rangle$ of S over U , we have $f_A \diamond f_A = f_A$.

Proof. Let $f_A = \langle (f_A^+, f_A^-); A \rangle$, be a DFS ideal of S over U . Then by Theorem 44, $f_A = f_A \cap f_A = f_A \diamond f_A$. Conversely, assume that $f_A = f_A \diamond f_A$ for each DFS ideal $f_A = \langle (f_A^+, f_A^-); A \rangle$ of S over U . Let A be an ideal of S . Then the characteristic double-framed soft mapping X_A of A is a DFS ideal of S over U . By hypothesis, $X_A = X_A \diamond X_A$. By Lemma 19, we have $X_A \diamond X_A = X_{(A^2)}$. Hence, $X_A = X_{(A^2)}$ and $A = (A^2]$ (Lemma 20). Thus, S is semisimple (Lemma 43).

Definition 46 A DFS ideal $f_A = \langle (f_A^+, f_A^-); A \rangle$ of S over U is called an irreducible DFS ideal of S over U , if for any DFS ideal $h_C = \langle (h_C^+, h_C^-); C \rangle$ and $g_B = \langle (g_B^+, g_B^-); B \rangle$ of S over U such that $g_B \cap h_C = f_A$ implies $g_B = f_A$ or $h_C = f_A$. That is,

$$g_B^+ \cap h_C^+ \cong f_A^+ \text{ and } g_B^- \cup h_C^- \cong f_A^- \text{ imply } g_B^+ \cong f_A^+ \text{ or } h_C^+ \cong f_A^+ \text{ and } g_B^- \cong f_A^-, h_C^- \cong f_A^-.$$

Proposition 47 A DFS ideal $f_A = \langle (f_A^+, f_A^-); A \rangle$ of S over U is a prime DFS ideal if and only if $f_A = \langle (f_A^+, f_A^-); A \rangle$ is a semiprime and irreducible DFS ideal of S over U .

Proof. Let $f_A = \langle (f_A^+, f_A^-); A \rangle$ be a prime DFS ideal of S over U . Let $g_B = \langle (g_B^+, g_B^-); B \rangle$ and $h_C = \langle (h_C^+, h_C^-); C \rangle$ be any DFS ideals of S over U such that $g_B \cap h_C = f_A$. Since $g_B \diamond h_C \subseteq g_B \cap h_C = f_A$ and $f_A = f_A$, A is prime DFS ideal over U , hence we have $g_B \subseteq f_A$ or $h_C \subseteq f_A$. On the other hand, $g_B \cap h_C = f_A$ implies $f_A \subseteq g_B$ or $f_A \subseteq h_C$. Hence, $g_B = f_A$ or $h_C = f_A$.

Conversely, assume that $f_A = \langle (f_A^+, f_A^-); A \rangle$ be an irreducible semiprime DFS ideal of S over U . Let $g_B = \langle (g_B^+, g_B^-); B \rangle$ and $h_C = \langle (h_C^+, h_C^-); C \rangle$ be any DFS ideals of S over U such that $g_B \cap h_C \subseteq f_A$. Since $(g_B \cap h_C) \diamond (g_B \cap h_C) \subseteq (g_B \cap h_C) \subseteq f_A$ and f_A is a semiprime DFS ideal, we have $(g_B \cap h_C) \subseteq f_A$. By Lemma 25, we have $f_A \cup (g_B \cap h_C) = (f_A \cup g_B) \cap (f_A \cup h_C)$. But $f_A \cup (g_B \cap h_C) = f_A$. Thus, $(f_A \cup g_B) \cap (f_A \cup h_C) = f_A$. Since f_A is irreducible, so we have $(f_A \cup g_B) = f_A$ or $(f_A \cup h_C) = f_A$. Hence $g_B \subseteq f_A$ or $h_C \subseteq f_A$. Thus, $f_A = \langle (f_A^+, f_A^-); A \rangle$ is a prime DFS ideal of S over U .

Proposition 48 Let $f_A = \langle (f_A^+, f_A^-); A \rangle$ be a DFS ideal of S over U with $f_A^+(a) = \gamma$ and $f_A^-(a) = \delta$, where $a \in S$ and $\gamma, \delta \in U$. Then there exists an irreducible DFS ideal $g_B = \langle (g_B^+, g_B^-); B \rangle$ of S over U such that $f_A \subseteq g_B$ and $f_A^+(a) = \gamma$ and $f_A^-(a) = \delta$.

Proof. Let $X := \{ \langle (g_B^+, g_B^-); B \rangle : \langle (g_B^+, g_B^-); B \rangle \text{ is a DFS ideal of } S \text{ over } U, \begin{cases} g_B^+(a) = \gamma, g_B^-(a) = \delta \text{ and } f_A^+ \subseteq g_B^+, f_A^- \subseteq g_B^- \end{cases} \}$

Then $X \neq \emptyset$, since $\langle (f_A^+, f_A^-); A \rangle \in X$. The collection X is partially ordered set by set inclusion. Suppose that Y is a totally ordered subset of X , say $Y := \{ \langle (g_{B_i}^+, g_{B_i}^-); B_i \rangle : i \in I \}$. Then,

$\langle (\bigcup_{i \in I} g_{B_i}^+, \bigcap_{i \in I} g_{B_i}^-); B \rangle$ is a DFS ideal of S over U . Since $f_A^+ \subseteq g_{B_i}^+$ and $f_A^- \subseteq g_{B_i}^-$ for each $i \in I$, $f_A^+ \subseteq \bigcup_{i \in I} g_{B_i}^+$ and $f_A^- \subseteq \bigcap_{i \in I} g_{B_i}^-$. Also $(\bigcup_{i \in I} g_{B_i}^+)(a) = \bigcup_{i \in I} (g_{B_i}^+(a)) = \gamma$ and $(\bigcap_{i \in I} g_{B_i}^-)(a) = \bigcap_{i \in I} (g_{B_i}^-(a)) = \delta$. Thus, $\bigcup_{i \in I} g_{B_i}^+$ is the least upper bound of Y , by Zorn's Lemma, there exists a DFS ideal $\langle (g_B^+, g_B^-); B \rangle$ of S over U which is maximal with respect to the property that $f_A^+ \subseteq g_B^+$, $f_A^- \subseteq g_B^-$ and $g_B^+(a) = \gamma$, $g_B^-(a) = \delta$.

Now we show that $\langle (g_B^+, g_B^-); B \rangle$ is irreducible FDS ideal of S over U . Suppose that $g_B^+ \cong g_{B_1}^+ \cap g_{B_2}^+$ and $g_B^- \cong g_{B_1}^- \cap g_{B_2}^-$, where $g_{B_1}^+ = \langle (g_{B_1}^+, g_{B_1}^-); B_1 \rangle$ and $g_{B_2}^+ = \langle (g_{B_2}^+, g_{B_2}^-); B_2 \rangle$ are DFS ideals of S over U . We claim that $g_B^+ \cong g_{B_1}^+$ or $g_B^+ \cong g_{B_2}^+$. Suppose on contrary that $g_B^+ \not\cong g_{B_1}^+$ and $g_B^+ \not\cong g_{B_2}^+$. Since $\langle (g_B^+, g_B^-); B \rangle$ is maximal with respect to the property that $g_B^+(a) = \gamma$, $g_B^-(a) = \delta$ and $g_B^+ \not\cong g_{B_1}^+$, $g_B^+ \not\cong g_{B_2}^+$ and $g_B^- \not\cong g_{B_1}^-$, $g_B^- \not\cong g_{B_2}^-$, it follows that $g_{B_1}^+(a) \not\cong \gamma \not\cong g_{B_2}^+(a)$ and $g_{B_1}^-(a) \not\cong \delta \not\cong g_{B_2}^-(a)$. Hence $\gamma = g_{B_1}^+(a) \cap g_{B_2}^+(a) = (g_{B_1}^+ \cap g_{B_2}^+)(a) \not\cong \gamma$ and $\delta = g_{B_1}^-(a) \cup g_{B_2}^-(a) = (g_{B_1}^- \cup g_{B_2}^-)(a) \not\cong \delta$, a contradiction. Hence either $g_B^+ \cong g_{B_1}^+$ or $g_B^+ \cong g_{B_2}^+$. Therefore, $\langle (g_B^+, g_B^-); B \rangle$ is an irreducible DFS ideal of S over U .

Theorem 49 Let $(S, \leq)$ be an ordered semigroup. Then the following conditions are equivalent.

- (1) S is semisimple.
- (2) $f_A \diamond g_B = f_A \cap g_B$, for all DFS ideals $f_A = \langle (f_A^+, f_A^-); A \rangle$ and $g_B = \langle (g_B^+, g_B^-); B \rangle$ of S over U .
- (3) $f_A \diamond f_A = f_A$, for every DFS ideal $f_A = \langle (f_A^+, f_A^-); A \rangle$ of S over U .
- (4) Every DFS ideal of S over U is semiprime DFS ideal of S over U .
- (5) Each DFS ideal of S over U is the intersection of prime DFS ideals of S over U .

Proof. (1) $\Leftrightarrow$ (2). Follows from Theorem 44.

(1) $\Leftrightarrow$ (3). Follows from Corollary 45.

(3) $\Rightarrow$ (4). Let $f_A = \langle (f_A^+, f_A^-); A \rangle$ and $g_B = \langle (g_B^+, g_B^-); B \rangle$ be any two DFS ideals of S over U such that $f_A \diamond f_A \subseteq g_B$. By hypothesis, $f_A \diamond f_A = f_A$, hence $f_A \subseteq g_B$. Hence each DFS ideal of S over U is semiprime.

(4) $\Rightarrow$ (5). Let $f_A = \langle (f_A^+, f_A^-); A \rangle$ be a proper DFS ideal and $\{ f_{A_i} = \langle (f_{A_i}^+, f_{A_i}^-); A_i \rangle : i \in I \}$ be the collection of all irreducible DFS ideals of S over U which contains

$f_A = \langle (f_A^+, f_A^-); A \rangle$. Proposition 48, guarantees the existence of such DFS ideals. Hence $f_A \subseteq \bigcap_{i \in I} f_{A_i}$. Let $x \in S$. Then by Proposition 48, there exists an irreducible DFS ideal $f_{A_\alpha} = \langle (f_{A_\alpha}^+, f_{A_\alpha}^-); A_\alpha \rangle$ of S over U such that $f_A \subseteq f_{A_\alpha}$ and $f_{A_\alpha}^+(x) = f_A^+(x)$ and $f_{A_\alpha}^-(x) = f_A^-(x)$. Thus, f_{A_α}

$\in \{ f_{A_i} = \langle (f_{A_i}^+, f_{A_i}^-); A_i \rangle : i \in I \}$. Hence $\bigcap_{i \in I} f_{A_i} \subseteq f_{A_\alpha}$. So $\bigcap_{i \in I} f_{A_i}^+(x) \subseteq f_{A_\alpha}^+(x) = f_A^+(x)$ and $\bigcap_{i \in I} f_{A_i}^-(x) \subseteq f_{A_\alpha}^-(x) = f_A^-(x)$. Thus, $\bigcap_{i \in I} f_{A_i} \subseteq f_A$. Consequently, $\bigcap_{i \in I} f_{A_i} = f_A$. By hypothesis, each DFS ideal over U is a semiprime DFS ideal over U , so each DFS ideal of S over U is the intersection of all irreducible semiprime DFS ideals of S over U which contain it. Also, each irreducible semiprime DFS ideal of S over U is prime. Therefore, each DFS ideal is the intersection of all prime DFS ideals of S over U which contain it.

(5) $\Rightarrow$ (3). Let $f_A = \langle (f_A^+, f_A^-); A \rangle$ be a proper DFS ideal of S over U . Then $f_A \diamond f_A$ is a DFS ideal over U . Since f_A is a DFS ideal of S over U , so $f_A \diamond f_A \subseteq f_A$. By hypothesis, we have $f_A \diamond f_A = \bigcap_{i \in I} f_{A_i}$, where $f_{A_i} = \langle (f_{A_i}^+, f_{A_i}^-); A_i \rangle$ are prime DFS ideals of S over U . Thus, $f_A \diamond f_A = f_{A_i}$ for all $i \in I$. Hence $f_A \subseteq f_{A_i}$ for all $i \in I$. Thus, $f_A \subseteq \bigcap_{i \in I} f_{A_i} = f_A \diamond f_A$. Hence $f_A \diamond f_A = f_A$.

Proposition 50 Let S be a semisimple ordered semigroup and $f_A = \langle (f_A^+, f_A^-); A \rangle$ be a DFS ideal of S over U . Then the following are equivalent.

- (1) $f_A = \langle (f_A^+, f_A^-); A \rangle$ is a prime DFS ideal of S over U .

(2) $f_A = \langle (f_A^+, f_A^-); A \rangle$ is an irreducible DFS ideal of S over U .

Proof. (1) $\Rightarrow$ (2). Suppose that $f_A = \langle (f_A^+, f_A^-); A \rangle$ is a prime DFS ideal of S over U . Let $g_B = \langle (g_B^+, g_B^-); B \rangle$ and $h_C = \langle (h_C^+, h_C^-); C \rangle$ be any two DFS ideals of S over U such that $g_B \cap h_C \subseteq f_A$. Since $g_B \cap h_C \subseteq g_B \cap h_C \subseteq f_A$. So either $g_B \subseteq f_A$ or $h_C \subseteq f_A$. On the other hand, $g_B \cap h_C = f_A$ implies $f_A \subseteq g_B$ and $f_A \subseteq h_C$. Thus, $f_A = g_B$ and $f_A = h_C$.

(2) $\Rightarrow$ (1). Suppose that $f_A = \langle (f_A^+, f_A^-); A \rangle$ is an irreducible DFS ideal of S over U . Let $g_B = \langle (g_B^+, g_B^-); B \rangle$ and $h_C = \langle (h_C^+, h_C^-); C \rangle$ be any two DFS ideals of S over U such that $g_B \cap h_C \subseteq f_A$. Since S is semisimple, Theorem 44, implies that $g_B \cap h_C = g_B \cap h_C$. Hence $g_B \cap h_C \subseteq f_A$. Thus, $(g_B \cap h_C) \cup f_A = f_A$. But $(g_B \cap h_C) \cup f_A = (g_B \cup f_A) \cap (h_C \cup f_A)$. Hence $(g_B \cup f_A) \cap (h_C \cup f_A) = f_A$. Since f_A is an irreducible DFS ideal of S over U , we have $g_B \cup f_A = f_A$ or $h_C \cup f_A = f_A$. Therefore, either $g_B \subseteq f_A$ or $h_C \subseteq f_A$.

Theorem 51 Let S be an ordered semigroup. Then every DFS ideal of S over U is prime DFS ideal of S over U if and only if the set of all DFS ideals of S over U forms a chain and S is semisimple.

Proof. Suppose that every DFS ideal of S over U is a prime DFS ideal and hence is semiprime. Then by Theorem 44, S is semisimple. Let $f_A = \langle (f_A^+, f_A^-); A \rangle$ and $g_B = \langle (g_B^+, g_B^-); B \rangle$ be any two DFS ideals of S over U . Then $f_A \cap g_B$ is a DFS ideal of S over U . By hypothesis, $f_A \cap g_B$ is a prime DFS ideal of S over U . Since $f_A \cap g_B \subseteq f_A \cap g_B$, so $f_A \subseteq f_A \cap g_B$ or $g_B \subseteq f_A \cap g_B$. Hence $f_A \subseteq g_B$ or $g_B \subseteq f_A$. Conversely, assume that S is semisimple and the set of all DFS ideals of S over U is totally ordered under inclusion. Let $f_A = \langle (f_A^+, f_A^-); A \rangle$, $g_B = \langle (g_B^+, g_B^-); B \rangle$ and $h_C = \langle (h_C^+, h_C^-); C \rangle$ be any DFS ideals of S over U such that $g_B \cap h_C \subseteq f_A$. Since the set of DFS ideals of S over U is totally ordered. So $g_B \subseteq h_C$ or $h_C \subseteq g_B$. If $g_B \subseteq h_C$, then $g_B = g_B \cap h_C \subseteq g_B \cap h_C \subseteq f_A$. If $h_C \subseteq g_B$, then $h_C = h_C \cap h_C \subseteq g_B \cap h_C \subseteq f_A$. Therefore, every DFS ideal of S over U is prime.

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